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# Practical Issues Associated with Mortar Projections in Large Deformation Contact/Impact Analysis

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**Abstract**
This paper is concerned with issues involved in the use of mortar projection methods to describe contact-impact phenomena in large deformation contact analysis. It considers issues pertaining to contact patch test passage, implementational details associated with numerical integration of the mortar projection integrals, and alternatives for mortar multiplier discretization.

## 1 Introduction

Several recent works (see for example, [1], [2], [3]) have considered variants of the mortar-finite element method for numerical treatment of contact phenomena. The method has shown considerable promise for the spatial discretization of contact interactions, particularly for kinematically linear applications where one or both of the contacting surfaces are flat. Desirable features already demonstrated for the method in this specialized setting include passage of patch tests, preservation of convergence rates that would be obtained with a perfectly conforming mesh, and accurate resolution of contact stresses on interfaces.
This paper concerns itself with the successful extension of these methods to encompass contact of geometrically noncoincident surfaces. The issue of patch test passage over curved interfaces will be discussed. It will be shown that a generalization of the mortar projection method is required to pass patch tests in this instance. Issues relating to the exact numerical integration of the mortar projection integrals will also be outlined, and a convergence study for a mortar tying application will be presented.

## 2 Mortar Formulation of the Mesh Tying Problem

Although the focus of this paper is contact problems in large deformations, the implementational details on which this paper concentrates can be more compactly discussed in the context of the mesh tying problem in small strains. Extension of these concepts to the treatment of unilateral contact constraints in large deformations is straightforward, and has already been demonstrated for the infinitesimal case (for example) in [1] and [2]. In giving a conceptual overview of the mesh tying problem, we consider two bodies $\Omega^{(i)} \in \mathbb{R}^3$, $i = 1, 2$, over which displacement fields $\mathbf{u}^{(i)} \in H^1(\Omega^{(i)})$, $i = 1, 2$ are to be found. We presume that the boundaries $\partial\Omega^{(i)}$ of the two bodies may each be decomposed via

$$\partial\Omega^{(i)} = \Gamma_\sigma^{(i)} \cup \Gamma_u^{(i)} \cup \Gamma_c^{(i)} \tag{1}$$

such that the surfaces $\Gamma_\sigma^{(i)}$, $\Gamma_u^{(i)}$ and $\Gamma_c^{(i)}$ satisfy

$$\Gamma_\sigma^{(i)} \cap \Gamma_u^{(i)} = \Gamma_u^{(i)} \cap \Gamma_c^{(i)} = \Gamma_\sigma^{(i)} \cap \Gamma_c^{(i)} = \emptyset. \tag{2}$$

The regions $\Gamma_\sigma^{(i)}$, $\Gamma_u^{(i)}$ and $\Gamma_c^{(i)}$ denote subsets of $\partial\Omega^{(i)}$ where traction, displacement, and mesh tying constraints will be imposed, respectively. The traction and displacement boundary conditions can be stated in the standard manner as

$$\begin{aligned} \sigma_{ij}^{(i)} n_j^{(i)} &= \bar{t}_i^{(i)} \text{ on } \Gamma_\sigma^{(i)} \\ u_i^{(i)} &= \bar{u}_i^{(i)} \text{ on } \Gamma_u^{(i)} \end{aligned} \tag{3}$$

where $\bar{t}_i^{(i)}$ and $\bar{u}_i^{(i)}$ denote the prescribed tractions and displacements associated with body $(i)$. In (3) and in what follows, indicial notation is assumed except where noted over spatial indices $i, j, k, l$ (not to be confused with body indices $(i)$).

In a typical mortar approach to this problem, one would introduce an additional field $\lambda : \Gamma_c^{(1)} = \Gamma_c^{(2)} \to \mathbb{R}^3$ over the mesh tying region, and render the following Lagrangian functional stationary with respect to the variables $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, and $\lambda$:

$$\Pi^{lagr}(\mathbf{u}, \lambda) := \Pi^{int,ext}(\mathbf{u}) - \int_{\Gamma_c^{(1)}} \lambda \cdot (\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\, d\Gamma \tag{4}$$

where the notation $\mathbf{u} = \{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}\}$ has been used to indicate the collection of the mappings over the two bodies (a similar notation will be used in the sequel for weighting functions, via $\mathbf{w} = \{\mathbf{w}^{(1)}, \mathbf{w}^{(2)}\}$). In (4), $\Pi^{int,ext}$ indicates the sum of potential energies of the two bodies to be mechanically tied.

We will assume for simplicity that the total virtual work for the two bodies, $G^{int,ext}(\mathbf{u},\mathbf{w}) = G^{int,ext^{(1)}}(\mathbf{u},\mathbf{w}) + G^{int,ext^{(2)}}(\mathbf{u},\mathbf{w})$, is derivable from the energy functional $\Pi^{int,ext}(\mathbf{u},\mathbf{w})$ via

$$G^{int,ext}(\mathbf{u},\mathbf{w}) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Pi^{int,ext}(\mathbf{u}+\epsilon\mathbf{w}). \tag{5}$$

Using a classical argument, rendering (4) stationary with respect to $\{\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \lambda\}$ gives

$$\begin{aligned} 0 &= \left.\tfrac{d}{d\epsilon}\right|_{\epsilon=0} \Pi^{lagr}(\mathbf{u}+\epsilon\mathbf{w},\lambda) \\ &= G^{int,ext}(\mathbf{u},\mathbf{w}) - \int_{\Gamma_c^{(1)}} \lambda\cdot(\mathbf{w}^{(1)}-\mathbf{w}^{(2)})\,d\Gamma \end{aligned} \tag{6}$$

which must hold for all $\{\mathbf{w}^{(1)}, \mathbf{w}^{(2)}\} \in \{\mathcal{V}^{(1)}, \mathcal{V}^{(2)}\}$, and

$$\begin{aligned} 0 &= \left.\tfrac{d}{d\epsilon}\right|_{\epsilon=0} \Pi^{lagr}(\mathbf{u},\lambda+\epsilon\mathbf{q}) \\ &= \int_{\Gamma_c^{(1)}} \mathbf{q}\cdot(\mathbf{u}^{(1)}-\mathbf{u}^{(2)})\,d\Gamma \end{aligned} \tag{7}$$

which must hold for all $\mathbf{q} \in \mathcal{M}$, where $\mathcal{M}$ is the space of admissible variations of the multiplier field $\lambda$. Equation (7) makes it clear that the mortar method has the effect of enforcing displacement continuity in an integral fashion over the interface between the bodies; in the case where unilateral constraints were involved, it would be the impenetrability and (if appropriate) frictional conditions that would be subject to such integral (rather than pointwise) treatment.

The traditional definition of a mortar discretization (see [4]) considers the following definition of the discrete multiplier space $M^h$, which acts to weakly enforce compability constraints on $\Gamma_c^{(1)^h}$ involving $\mathbf{u}^h$, which is assumed here to be interpolated via piecewise polynomials of order $m$: [1]

$$M^h = \left\{ \mathbf{q}^h \left| \begin{array}{l} \mathbf{q}^h|_{\Gamma_c^{(1)^{h^e}}} \in \mathbb{P}^m(\Gamma_c^{(1)^{h^e}}),\ \text{if}\ \bar{\Gamma}_c^{(1)^{h^e}} \cap \partial\Gamma_c^{(1)^h} = \emptyset \\ \mathbf{q}^h|_{\Gamma_c^{(1)^{h^e}}} \in \mathbb{P}^{m-1}(\Gamma_c^{(1)^{h^e}}),\ \text{if}\ \bar{\Gamma}_c^{(1)^{h^e}} \cap \partial\Gamma_c^{(1)^h} \neq \emptyset \end{array} \right. \right\} \tag{8}$$

In words, the weighting functions $\mathbf{q}^h$ are interpolated at the same order as the solution $\mathbf{u}^h$ on elements on the interior of mortar surfaces, and at one order less for elements on the boundaries of these surfaces.

## 3 Implementation of Mortar Projections on Noncoincident Interfaces

The presentation given above ignores an important practical detail in most contact or mesh tying applications; namely, that even if the surfaces $\Gamma_c^{(i)}$ are geometrically coincident before discretization, the discretized counterparts $\Gamma_c^{(i)^h}$ will not in general be the same if different meshing strategies are used in the two regions to be joined (see Figure 1).

Indeed, most of the literature existing for mortar methods considers the case where the surfaces to be joined are geometrically coincident, which is to say that most often they are flat. When they are not

[1]The original proposal in [4] included imposition of pointwise continuity on the edges of the surface. Such extra constraints are now known to be unnecessary for optimal convergence results.

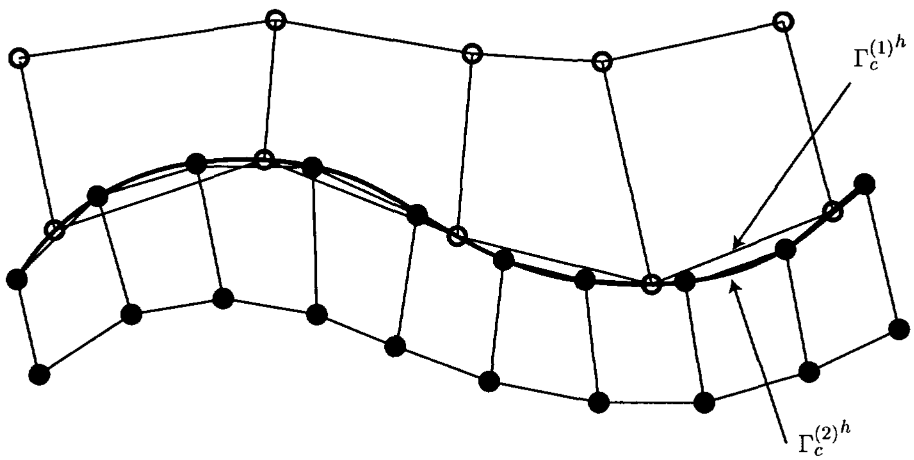


Figure 1: Simple two dimensional illustration of how dissimilar polynomial interpolation of the same surface causes $\Gamma_c^{(1)^h} \neq \Gamma_c^{(2)^h}$. The dark curve indicates the surface to be approximated.

flat and/or coincident, as depicted in Figure 1, at least two issues become prominent in the use of these methods:

- Since $\Gamma_c^{(i)}$, $i = 1, 2$ differ, one must decide in general which surface will serve as the domain of integration for the mortar projection integrals (i.e., those in (7)). Furthermore, one needs an algorithm to compute the intersections of the two tied surfaces, such that each intersection (or smooth segment) corresponds to only one element surface from each side of the interface. A procedure for performing this computation is summarized in Subsection 3.1
- When the two surfaces are noncoincident, the issue of patch test passage (trivial for a mortared flat interface) becomes a significant issue. In general, unmodified mortar methods do not pass patch tests for noncoincident domains. This issue is discussed further in Subsection 3.2.

### 3.1 Numerical Evaluation of Mortar Projection Integrals

The problem of surface integration is depicted schematically in Figure 2. The strategy used, described in more detail in [5], is to first divide the domain of integration for the mortar integrals into subdomains or segments, each of which involves only one element surface from each side of the interface.

A procedure for calculating the mortar integrals may then be summarized as follows:

1. Loop over elements $k$ on non-mortar side (i.e., side of the interface where the multipliers are defined)
   (a) Loop over mortar side elements $l$
      i. Perform rough screen to determine if facet $l$ is close to facet $k$
      ii. If $l$ "far" from $k$, go to top of mortar element loop and increment $l$
      iii. Form polygon representing overlap of facets $l$ and $k$

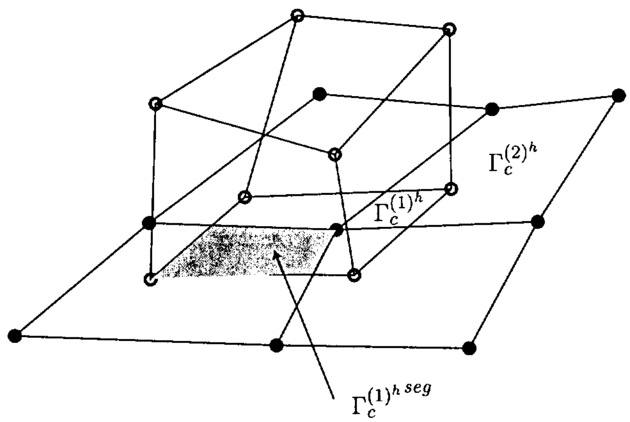


Figure 2: Illustration of the segment concept in three dimensions; each segment consists of only one element surface on each side of the interface. Shaded area denotes one surface segment.

iv. Locate geometric center of polygon and divide polygon up into $np$ triangular pallets.
v. Use Gauss-Radau rules (see, e.g., [6]) to locate $ng$ Gauss points $\mathbf{x}_g$ on each triangular pallet.
vi. Identify Gauss points $\mathbf{x}_g$ with points on facets $k$ and $l$ to get parametric local coordinates $\xi_{g,k}$ and $\xi_{g,l}$
vii. Compute contributions to inner products $\int N_A N_B \, d\Gamma$ over pallet $p$ for $i = 1, 2$ for all relevant $A$ and $B$ shape functions defined over facets.
viii. Assemble mortar contributions by summing contributions over all pallets

(b) End loop over mortar side elements

2. End loop over non-mortar side elements

As a result of the above procedure, one ends up with a triangulated "quadrature mesh" which represents the imprint of one surface on the other. We therefore term this procedure "mesh imprinting;" an example intersection of two dissimilar meshes (including the triangular integration pallets referred to in step iv. above) is depicted in Figure 3.

### 3.2 Patch Tests and Consistent Mesh Tying

In contact mechanics, the concept of a patch test is rather straightforward to describe conceptually: we demand that a contact interface be able to exactly transmit a spatially constant pressure field from one body to another when equilibrium conditions dictate that it should. For tied contact, this requirement changes slightly, such that a numerical description containing a tied interface should be able to exactly represent arbitrary spatially constant stress states. As mentioned previously, a geometrically coincident mortar implementation is capable of passing such patch tests. However, for noncoincident surfaces $\Gamma_c^{(1)^h}$ and $\Gamma_c^{(2)^h}$ joined together in the manner described above, such spatially constant stress fields *will not* in general be represented correctly. This occurs essentially due to two reasons:

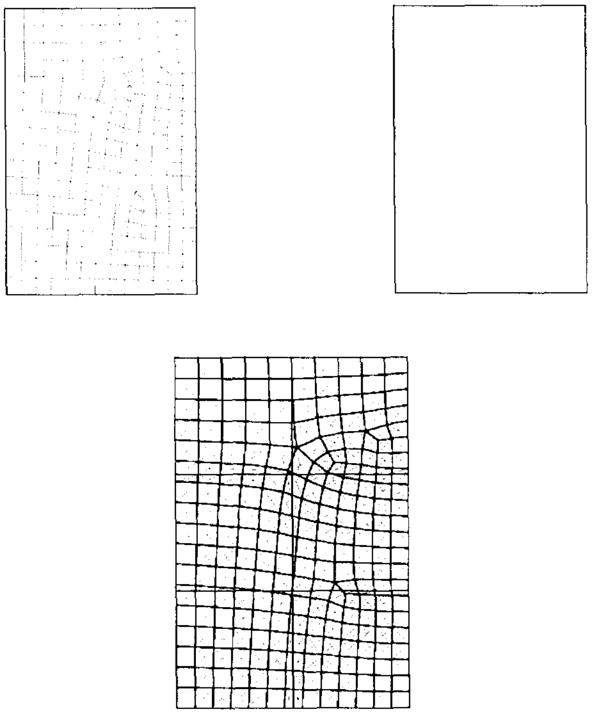

Figure 3: Simple tension bar example depicting mesh imprinting algorithm. Bottom diagram is the triangulated intersection of the two meshes.

1. Mortar methods employ multipliers which are *tractions*, which do not contain enough information to describe the same stress state contracted into two distinct normals (such as are featured by the two sides of a noncoincident interface); and
2. Noncoincident interfaces give rise to regions of overlap or void between the two volumes being joined, which in should in turn have *strain energy* associated with them in a constant stress patch test. Since a mortar method has no way to describe such strain energy (in fact, it seeks to make it zero in a weak sense), it effectively constrains out the kinematics necessary to pass a patch test.

In response to these concerns, Laursen and Heinstein ([7]) recently proposed a consistent mesh tying formulation as a generalization of the mortar concept, in which three fields are considered (displacements, interface stresses, and interface strains) in a Hu-Washizu type of approach. Specifically, one renders the following functional stationary:

$$\begin{aligned} &\Pi^{HW^h}(\mathbf{u}^h, \sigma^h, \epsilon^h) := \Pi^{int,ext^h}(\mathbf{u}^h) \\ &- \sum_{seg=1}^{n_{seg}} \left[ \int_{\Omega_m^{h^{seg}}} i_{vol}[\sigma^h : (\epsilon^h - \nabla \mathbf{u}^{m^h}) - \tfrac{1}{2}\epsilon^h : \mathbf{C}\epsilon^h]\, d\Omega \right] \end{aligned} \qquad (9)$$

with respect to displacements $\mathbf{u}^h$, and interface stresses and strains $\sigma^h$ and $\epsilon^h$, which are defined over volumes of mortar segments $\Omega_m^{h^{seg}}$. In [7], this formulation is demonstrated in a large deformation con-

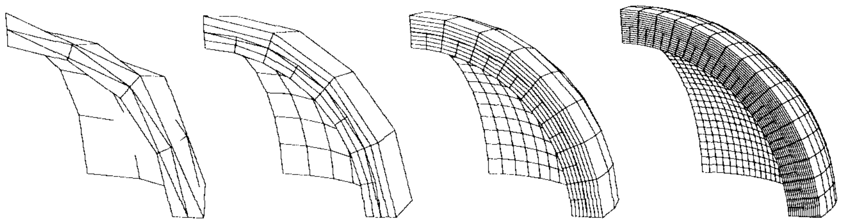

Figure 4: Meshes used for the pressurized sphere convergence study.

text within a matrix free, augmented Lagrangian equation solving framework, and is shown to facilitate exact patch test passage. However, in this reference, the method is formulated in conjunction with the use of constant stress continuum elements, which helps greatly in the development of an effective iterative scheme for the interface stresses. In a more general context, where modes may be present in the formulation and higher order interpolation of stresses is likely to be desirable, the effective use of such *consistent tying methods* remains an open question.

## 4 Conclusions and Summary

This paper has dealt with mortar projection issues associated with contact applications in which contacting surfaces are geometrically noncoincident (i.e., in general, not flat). It has been seen that the mortar integrals involved in achieving the projection may be straightforwardly and robustly calculated. However, it has also been pointed out that such mortar projection schemes will not in general produce passage of patch tests. To this end, a generalization of mortar projection we term *consistent mesh tying* has been introduced.

Our current work is focused on numerical investigation of whether such patch test passage is necessary for optimal convergence rates to be obtained in contact problems. Early indications are that it is not. As an example of this assertion, we refer to Figure 4, which depicts the meshes used in a convergence study involving a pressurized sphere. As can be seen, the meshes on either side of the tied interface are obviously noncoincident, and as a result do not pass patch tests as discussed above. However, consultation of Figure 5 clearly shows that optimal convergence is obtained in the energy norm, whereas nodal collocation (corresponding to traditional node-to-surface tying) suffers a noticeable degradation in convergence. Results are shown for runs on which the multipliers are discretized on the course mesh, and also for runs where they are discretized on the fine mesh. Two different mortar implementations are depicted; the one corresponding to the discretization summarized in 8) (termed linear), and one corresponding to the *dual formulation* proposed by Wohlmuth (see, e.g., [3], termed dual). Optimal convergence is obtained in all cases, with the *magnitude* of the error virtually the same as that produced by a conforming discretization.

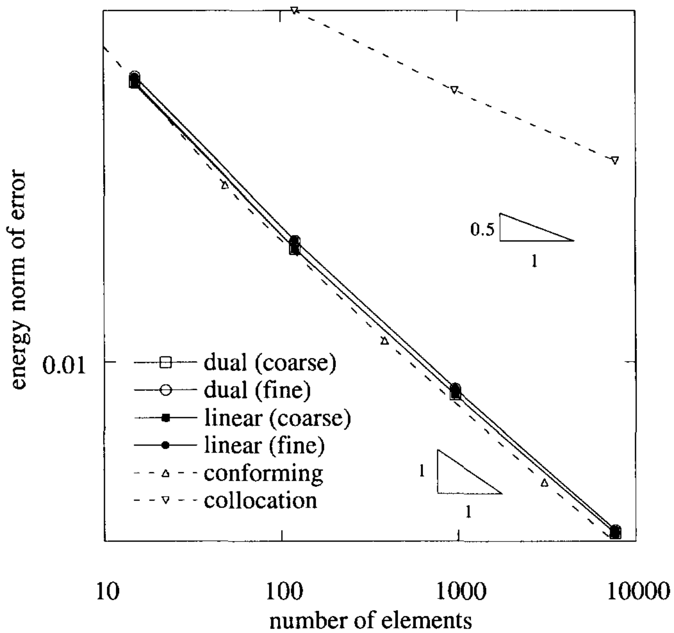


Figure 5: Error (as measured by the energy norm; optimal convergence rate is 1) in the pressurized sphere convergence study.

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